GENERALIZED MOYAL STRUCTURES IN PHASE-SPACE, KINETIC EQUATIONS AND THEIR CLASSICAL LIMIT II: APPLICATIONS TO HARMONIC OSCILLATOR MODELS

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ABSTRACT The formalism of generalized Wigner transformations developped in a previous paper, is applied to kinetic equations of the Lindblad type for quantum harmonic oscillator models. It is first applied to an oscillator coupled to an equilibrium chain of other oscillators having nearest-neighbour interactions. The kinetic equation is derived without using the so called rotating-wave approximation. Then it is shown that the classical limit of the corresponding phase-space equation is independent of the ordering of operators corresponding to the inverse of the generalized Wigner transformation, provided the latter is involutive. Moreover, this limit equation, which conserves the probabilistic nature of the distribution function and obeys an H-theorem, coincides with the kinetic equation for the corresponding classical system, which is derived independently and is distinct from that usually obtained in the litterature and not sharing the above properties. Finally the same formalism is applied to more general model equations used in quantum optics and it is shown that the above results remain unaltered.

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1. INTRODUCTION

In a previous paper ([1], here after called paper I), we studied the properties of generalized Weyl and Wigner transformations of quantum operators and classical phase-space functions respectively and with their help we presented a phase-space formulation of quantum evolution equations. In the case of open systems interacting with a large equilibrium bath we gave an explicit phase-space representation of quantum kinetic equations of the Lindblad type. As it is well-known, such equations are satisfactory in the sense that they conserve the probabilistic interpretation of the density matrix and obey an H-theorem ([2], [3], [4] section V, [5] proposition 4.3, [6] section 4.3.). They are obtained in the context of various approaches to kinetic theory that employ some more or less systematic approximation scheme to the exact dynamics of the system ([3]-[5], [7]-[12]). In the present paper we illustrate the formalism of paper I by applying it to kinetic equations of this kind, for particular harmonic oscillator models.

In section 2, we first apply the general kinetic equation for an open system weakly interacting with a bath at equilibrium, that follows in the context of the formal theory presented in [5], a brief account of which is given in Appendix 3, to the case of a *classical* harmonic oscillator interacting with a (linear) harmonic chain, assuming nearest neighbour interactions. Moreover, the chain is considered to be at canonical equilibrium. We show that in the thermodynamic limit of an infinite chain, the interaction Hamiltonian takes a simple separable form. The derivation uses functional methods and leads to an equation of the Fokker-Planck type in the *phase-space* of the oscillator.

Then in section 3 the corresponding *quantum* system is treated in the context of the same formalism, leading to a kinetic equation well-known in the litterature but either introduced phenomenologically or derived with the aid of additional approximations. Its phase-space representation via an arbitrary generalized Wigner transformation is accordingly given. It is also shown that *its classical limit* is *unique*, as long as the corresponding generalized Moyal bracket is a deformation of the Poisson bracket, and *coincides with the classical equation obtained in section 2*, which for the sake of completeness was derived there independently, thus exhibiting the consistency of our formalism, at least in this special case.

Finally in section 4, a more general model equation of the Lindblad type for a harmonic oscillator linearly coupled to a bath of other oscillators, that has been considered in the litterature especially in quantum optics ([13], [14] and references there in) is phase-space transformed with the aid of the formalism of paper I. This generalizes previous results of [14] and easily implies that once again *its classical limit is unique* under the condition stated above.

2. CLASSICAL OSCILLATOR LINEARLY COUPLED TO A HARMONIC CHAIN

In this section we will study a classical harmonic oscillator model and in the next section our results will be related to those of the corresponding quantum system. The model consists of a harmonic oscillator (the subsystem Σ) coupled to an *infinite* set of harmonic oscillators (the reservoir R), at canonical equilibrium. It is assumed that the interaction is (bi)linear in the coordinates of Σ and R. The Hamiltonian reads

$$H = H_\Sigma + H_R + \lambda' H_I \tag{2.1}$$

$$H_\Sigma = \frac{p'^2}{2M} + \frac{1}{2} k q'^2 \tag{2.2a}$$

$$H_R = \frac{1}{2m} \sum_k p_k'^{\,2} + \frac{1}{2} \sum_{k,l} q_k' h_{k-l}' q_l' \tag{2.2b}$$

$$H_I = \sum_k \epsilon_k' q_k' q' \tag{2.2c}$$

where λ' is a coupling parameter and the remaining symbols have an obvious meaning. Performing the (canonical) scale transformation

$$p = M^{-\frac{1}{2}} p' \qquad q = M^{\frac{1}{2}} q'$$

$$p_k = m^{-\frac{1}{2}} p_k' \qquad q_k = m^{\frac{1}{2}} q_k'$$

we set

$$H = H_\Sigma + H_R + \lambda H_I, \qquad \lambda = \sqrt{\frac{m}{M}} \lambda' \tag{2.3}$$

$$H_\Sigma = \frac{p^2}{2} + \Omega_0^2 q^2, \qquad \Omega_0^2 = \frac{k}{M} \tag{2.4a}$$

$$H_R = \frac{1}{2} \sum_k p_k^2 + \frac{1}{2} \sum_{k,l} q_k h_{k-l} q_l, \qquad h_k = \frac{h_k'}{m} \tag{2.4b}$$

$$H_I = \sum_k \epsilon_k q_k q \qquad \epsilon_k = \frac{\epsilon_k'}{m} \tag{2.4c}$$

Actually in defining H_R we may take any positive-definite symmetric matrix h_{kl} and not only a codiagonal one. This is readily seen for a *finite* chain by

taking an orthogonal matrix γ_{mn} diagonalizing h_{mn} and making the (canonical) transformation

$$(q'_k, p'_k) \rightarrow (\tilde{q}_k, \tilde{p}_k)$$

$$\tilde{p}_m = \sum_k \gamma_{km} p'_k, \qquad \tilde{q}_m = \sum_k \gamma_{mk} q'_k$$

However for an *infinite* system the diagonalization procedure is more involved since subtleties appear due to the (presummably simultaneous) existence of a continuous and a point spectrum of h_{kl}. Nevertheless, if it posseses a complete set of eigenvectors, normal coordinates can be defined in analogy with (2.6) below, hence (2.8) follows and consequently kinetic equation ((2.26) below) remains unmodified. In view of the above discussion it is reasonable to assume in the present case

$$h_k = h_k^* = h_{-k} \geq 0 \tag{2.5}$$

where h_k^* denotes the complex conjugate.

This model has been used extensively, particularly in the case of nearest neighbour interactions the various calculations being performed for a *finite* chain and then taking, the thermodynamic limit (see e.g. [15] eq(1), [16] eq(18), [17]-[19] and indirectly [20]). It should be noted that one often considers $\sqrt{\frac{m}{M}}$ as a small parameter (Brownian motion proper) and not just the coupling parameter λ' as in this paper.

In case of an infinite number of nonvanishing h_k's it will be assumed that $h_k \rightarrow 0$ as $\mid k \mid \rightarrow +\infty$ sufficiently rapidly and as a matter of fact, they will be defined as the coefficients of a periodic function. Normal coordinates are then introduced by the Fourier series

$$\phi(\theta) = \frac{1}{\sqrt{2\pi}} \sum_{k=-\infty}^{+\infty} q_k e^{ik\theta} \;\Leftrightarrow\; q_k = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} d\theta \phi(\theta) e^{-ik\theta} \tag{2.6a}$$

$$\pi(\theta) = \frac{1}{\sqrt{2\pi}} \sum_{k=-\infty}^{+\infty} p_k e^{-ik\theta} \;\Leftrightarrow\; p_k = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} d\theta \pi(\theta) e^{ik\theta} \tag{2.6b}$$

This transformation is canonical and $\xi_k(\theta) \equiv \frac{1}{\sqrt{2\pi}} e^{ik\theta}$ may be considered as a complete orthonormal set of vectors

$$\sum_{k,l} \xi_k^*(\theta) \xi_l(\theta') = \delta(\theta - \theta') \tag{2.7a}$$

$$\int_{-\pi}^{\pi} d\theta \xi_k(\theta) \xi_l^*(\theta) = \delta_{k,l} \tag{2.7b}$$

where $\delta(\theta)$ is periodic with period 2π and $\delta_{k,l}$ is the Kronecker delta. Introducing (2.6) in (2.4) we get

$$H_R = \frac{1}{2}\int_{-\pi}^{\pi} d\theta(|\pi(\theta)|^2 + \omega^2(\theta)|\phi(\theta)|^2) \tag{2.8a}$$

$$H_I = \int_{-\pi}^{\pi} d\theta u^*(\theta)\phi(\theta)q \tag{2.8b}$$

$$u(\theta) = \frac{1}{\sqrt{2\pi}}\sum_{n=-\infty}^{+\infty}\epsilon_n e^{in\theta} \quad \Leftrightarrow \quad \epsilon_n = \frac{1}{\sqrt{2\pi}}\int_{-\pi}^{\pi} d\theta u(\theta)e^{-in\theta} \tag{2.9}$$

$$\omega^2(\theta) \equiv \sum_{k=-\infty}^{+\infty} h_k e^{ik\theta} = \omega^2(-\theta) \geq 0 \tag{2.10}$$

because of (2.5). Note that ϵ_n is real and that it is reasonable to consider that $\epsilon_n = \epsilon_{-n}$ (cf. e.g. [15] eq.(10) for the finite case), so that $u(\theta)$ is real and even.

As we have used the basis $\{\frac{1}{\sqrt{2\pi}}e^{ik\theta}\}$, the variables $\phi(\theta), \pi(\theta)$ are complex-valued functions of θ. Had we chosen the basis $\{\frac{1}{\sqrt{2\pi}}, \frac{1}{\sqrt{\pi}}\cos k\theta, \frac{1}{\sqrt{\pi}}\sin k\theta, k \in \mathcal{N}\}$, the corresponding variables would be real. Note also that by (2.9), only the real part $\Re\phi(\theta)$ contributes to H_I in (2.8b).

In view of the above discussion, we will consider in the rest of this section a simple version of the Hamiltonian (2.8) namely

$$H_R = \frac{1}{2}\int d\omega(\pi^2(\omega) + \omega^2\phi^2(\omega)) \tag{2.11a}$$

$$H_I = \int d\omega u(\omega)\phi(\omega)q \equiv Wq \tag{2.11b}$$

Where $\pi(\omega), \phi(\omega)$ are real, *the intergration interval is symmetric about the origin and* $u(\omega)$ *is real and even* (this is not an essential assumption; see end of Appendix 1). It will become evident that results for the general case (2.8) follow by replacing $u(\omega)$ with $u^2(\omega)\frac{d\theta}{d\omega}$ in the final expressions (see also the discussion in Appendix 1).

Clearly Hamilton's equations for the total system are

$$\dot{q}(t) = p(t) \tag{2.12a}$$

$$\dot{p}(t) = -\Omega_0^2 q - \lambda\int d\omega u(\omega)\phi(\omega) \tag{2.12b}$$

$$\dot{\phi}(\omega) = \frac{\delta H}{\delta\pi(\omega)} = \pi(\omega) \tag{2.12c}$$

$$\dot{\pi}(\omega) = -\frac{\delta H}{\delta \phi(\omega)} = -\omega^2 \phi(\omega) - \lambda u(\omega) q \qquad (2.12d)$$

using functional derivatives. Because the Hamiltonian is quadratic and the reservoir in a canonical equilibrium state) functional methods (see e.g. [24],[25]) will be used in the following to treat ab initio the infinite system.

Because of its linearity, the model is exactly solvable, the solution being obtained by solving the generalized eigenvalue problem

$$\Omega_0^2 Z_\nu + \lambda \int d\omega' u(\omega') \xi_\nu(\omega) = \nu Z_\nu$$

$$\lambda u(\omega) Z_\nu + \omega^2 \xi_\nu(\omega) = \nu \xi_\nu(\omega)$$

For ν outside the interval ω^2 varies, we get

$$\xi_\nu(\omega) = -\lambda \frac{u(\omega)}{\omega^2 - \nu} Z_\nu$$

$$\left(\Omega_0^2 - \nu - \lambda^2 \int d\omega \frac{u^2(\omega)}{\omega^2 - \nu} \right) Z_\nu = 0$$

Thus for nontrivial solutions

$$\Omega_0^2 - \nu = \lambda^2 \int d\omega \frac{u^2(\omega)}{\omega^2 - \nu}$$

Nonnegativity of the Hamiltonian excludes negative values for ν which lead to instabilities, hence we must impose the constraint (cf. [15] eq.(24))

$$\Omega_0^2 \geq \lambda^2 \int d\omega \frac{u^2(\omega)}{\omega^2} \qquad (2.13)$$

which inter alia implies that at the origin $u^2(\omega)$ is $o(\omega^2)$ and if the range of ω is infinite $u^2(\omega)$ cannot grow at infinity more rapidly than $\mid \omega \mid^{2-\epsilon}$ for $0 < \epsilon < 1$. Clearly these considirations imply that Ω_0 cannot vanish, otherwise the model will necessarily lead to instabilities. Note however that, had we taken $H_I = \sum_k u_k (q_k - q)^2$ the problem would not arise as Ω_0^2 should be replaced by $\Omega_0^2 + \lambda \sum_k u_k$

Although the model is solvable in principle, we will consider weak coupling of Σ with R, and we will obtain the kinetic equation for the oscillator by employing the general formalism of [5] that was briefly discussed in section 4 of paper I and Appendix 3 here,that is, compute the generator Φ of Appendix

3. Because of the separability of H_I, eq.(2.11b), we can use the corresponding general formulas of [5] eqs (4.22), (4.22′). To this end we need

$$q(t) \equiv e^{iL_\Sigma t} q = q \cos \Omega_0 t + \frac{p}{\Omega_0} \sin \Omega_0 t \tag{2.14a}$$

$$W(t) \equiv e^{iL_R t} W \equiv \int d\omega \; u(\omega)\phi(\omega, t) =$$

$$\int d\omega \; u(\omega)(\phi(\omega) \cos \omega t + \frac{\pi(\omega)}{\omega} \sin \omega t) \tag{2.14b}$$

where we write $L_\Sigma = i\{H_\Sigma, \cdot\}$ etc for the various Liouville operators corresponding to (2.1) and we used that for $\lambda = 0$, (2.12) gives the solution $q(t), \phi(\omega, t)$. In the present case eqs (4.22), (4.22′) of [5] for the probability distribution f of the harmonic oscillator are (cf. (2.11b)):

$$\frac{\partial f}{\partial t} = \{H_\Sigma - \lambda^2 F, f\} +$$

$$+ \frac{\lambda^2}{2} \sum_n (\tilde{h}(\omega_n)\{(P_n q)^*, \{P_n q, f\}\} + \tilde{g}(\omega_n)\{(P_n q)^*, (P_n q) f\}) \tag{2.15}$$

$$F = \frac{1}{2} \sum_n (\bar{h}(\omega_n)\{(P_n q)^*, P_n q\} + \bar{g}(\omega_n)(P_n q)^* P_n q) \tag{2.16}$$

Here P_n, ω_n are the eigenprojections and eigenvalues of L_Σ,

$$\tilde{h}(\omega) = \int_{-\infty}^{+\infty} ds \; e^{i\omega s} h(s), \qquad h(s) \equiv < W \, W(s) > \tag{2.17a}$$

$$\tilde{g}(\omega) = \int_{-\infty}^{+\infty} ds \; e^{i\omega s} g(s), \qquad g(s) \equiv < W \;\, W(s) > \tag{2.17b}$$

$$\bar{h}(\omega) = \int_0^{+\infty} ds \; e^{i\omega s} h(s) \tag{2.17c}$$

$$\bar{g}(\omega) = \int_0^{+\infty} ds \; e^{i\omega s} g(s) \tag{2.17d}$$

and $< \quad >$ denotes the average over the chain, assumed in a canonical equilibrium state

$$\rho_R = \frac{1}{Z} e^{-\frac{\beta}{2} \int d\omega (\pi^2(\omega) + \omega^2 \phi^2(\omega)))} = \frac{1}{Z} e^{-\beta H_R} \tag{2.18a}$$

$$Z = \int \delta\phi\delta\pi \ e^{-\beta H_R} \tag{2.18b}$$

clearly

$$\omega_n = n\Omega_0, \qquad n \in \mathcal{Z}$$

$$P_n A(\theta, \zeta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} d\theta' e^{in(\theta - \theta')} A(\theta', \zeta) \tag{2.19}$$

with ζ, θ the action-angle variables of the oscillator:

$$q = \left(\frac{2\zeta}{\Omega_0}\right)^{\frac{1}{2}} \sin\theta, \qquad \frac{p}{\Omega_0} = \left(\frac{2\zeta}{\Omega_0}\right)^{\frac{1}{2}} \cos\theta \tag{2.20}$$

A simple calculation using (2.19), (2.20) yields

$$P_n q = \frac{1}{2}(q - \frac{ip}{\Omega_0})\delta_{n,1} + \frac{1}{2}(q + \frac{ip}{\Omega_0})\delta_{n,-1} \tag{2.21}$$

On the other hand explicit calculation of (2.17) involves certain simple functional integrals and derivatives. The details are given in Appendix 1. The results are:

$$\tilde{h}(\omega) = \frac{2\pi u^2(\omega)}{\beta\omega^2}, \qquad \tilde{g}(\omega) = i\beta\omega\tilde{h}(\omega) \tag{2.22}$$

$$\bar{h}(\omega) = \frac{\tilde{h}(\omega)}{2} + i \int d\omega' \frac{u^2(\omega')}{\beta\omega'} \frac{\omega}{\omega^2 - \omega'^2} \tag{2.23a}$$

$$\bar{g}(\omega) = \frac{\tilde{g}(\omega)}{2} - \int d\omega' \frac{u^2(\omega')}{\omega^2 - \omega'^2} \tag{2.23b}$$

Notice that because of (2.18), the second of (2.22) follows from the first and proposition 4.5 of [5]. An elementary calculation using (2.21), (2.22) gives dissipative part of (2.15)=

$$= \frac{\lambda^2\pi}{2\beta} \frac{u^2(\Omega_0)}{\Omega_0^2} \left[\frac{1}{\Omega_0^2} \frac{\partial^2 f}{\partial q^2} + \frac{\partial^2 f}{\partial p^2} + 2\beta(\frac{\partial}{\partial q}(qf) + \frac{\partial}{\partial p}(pf)) \right] \tag{2.24}$$

Similarly (2.21), (2.23) substituted in (2.16) give

$$\frac{1}{2} \sum_n \bar{h}(\omega_n) \{(P_n q)^*, P_n q\} = \frac{1}{2\beta} \int d\omega \frac{u^2(\omega)}{\omega^2} \frac{1}{\Omega_0^2 - \omega^2} = constant$$

$$\frac{1}{2}\sum_n \bar{g}(\omega_n)(P_n q)^* P_n q = -H_\Sigma \frac{1}{2\Omega_0^2}\int d\omega \frac{u^2(\omega)}{\Omega_0^2-\omega^2}$$

Since the first does not contribute to the first term on the r.h.s. of (2.15) we may write (2.16) as:

$$F = \frac{\Delta(\Omega_0)}{\Omega_0} H_\Sigma, \qquad \Delta(\Omega_0) = \frac{1}{2\Omega_0}\int d\omega \frac{u^2(\omega)}{\omega(\omega-\Omega_0)} \tag{2.25}$$

where we used that the range of ω is symmetric about the origin.By (2.24), (2.25), equation (2.15) takes the form

$$\frac{\partial f}{\partial t} = \left(1 - \frac{\lambda^2 \Delta(\Omega_0)}{\Omega_0}\right)\{H_\Sigma, f\} +$$

$$+\frac{\lambda^2\pi}{2\beta}\frac{u^2(\Omega_0)}{\Omega_0^2}\left(\frac{\partial}{\partial q}\left(\frac{1}{\Omega_0^2}\frac{\partial f}{\partial q}+2\beta q f\right)+\frac{\partial}{\partial p}\left(\frac{\partial f}{\partial p}+2\beta p f\right)\right) \tag{2.26}$$

we may notice that as expected, the Maxwell-Boltzman (MB) distribution $e^{-\beta H_\Sigma}$ is a stationary solution of (2.26) and F is an integral of the unperturbed motion, results that are special cases of proposition 4.1, 4.2 of the general formalism of [5] that led to (2.15).

It may also be remarked that in the litterature, a different kinetic equation has been derived for this model ([21] eq.(20) in connection with eq(31) of [15])

$$\frac{\partial f}{\partial t} - (1 - \frac{\lambda^2\Delta(\Omega_0)}{\Omega_0})\Omega_0^2 q \frac{\partial f}{\partial p} + p\frac{\partial f}{\partial q} =$$

$$\lambda^2 \frac{\partial}{\partial p}\left(\frac{\pi}{\beta}\frac{u^2(\Omega_0)}{\Omega_0^2}(\frac{\partial f}{\partial p} + 2\beta p f) + \frac{\chi(\Omega_0)}{\Omega_0}\frac{\partial f}{\partial q}\right)$$

$$\chi(\Omega_0) = \int d\omega \frac{u^2(\omega)}{\omega^2(\omega-\Omega_0)}$$

In fact it can be shown that this equation is the second order term in the λ-expansion of the so-called Generalized Master equation (GME), that follows by projecting the state of the total system $\Sigma + R$ to that of Σ, namely eqs. (3.7′) or (5.1) of [5] (see also Appendix 3). Clearly this equation has not a nonegative-definite 2nd order coefficient matrix and therefore it does not conserve the positivity of f and for that matter obeys no H-theorem. Moreover it does not have the MB distribution $ce^{-\beta H_\Sigma}$ as an equilibrium solution (cf. [5] sections 4, 5, [23] section 4, 5). A somewhat different equation but still with the structure of (2.27), hence the same undesirable features, has been obtained by a similar analysis of the GME, ([30],[31]; see also [34] and the discussion

in [5] section [5]). Sometimes the mixed derivatives-term is neglected arguing that it is "small" ([21] p.60) so that a conventional Fokker-Planck equation results. However even in this case the MB distribution is not stationary, that is even for a chain at canonical equilibrium, the external oscillator does not evolve toward such an equilibrium state! Moreover it should be noticed that in this case this neglect leads to different coefficients for the conventional part of the Fokker-Plank equation (compare (2.26), (2.27)). This is a general feature that can be understood on the basis of the general formalism presented in [5], since the markovian approximation of the GME leads to an equation ((3.7′) of [5] for the generator Θ in Appendix 3) totally different from that obtained by our formalism ((3.11′) of [5] for the generator Φ in Appendix 3). This fact has been verified in other models as well (e.g. compare eqs (4.1), (4.2) with (4.4), (4.5) of [33] section 4). A further argument in favor of (2.26) is provided in the next section, where we show that (2.26) is the unique classical limit of the kinetic equation of the Lindblad type for the corresponding quantum system, which is well-known and widely used, e.g. in quantum optics.

3. KINETIC EQUATION FOR A QUANTUM HARMONIC OSCILLATOR, WEAKLY-COUPLED TO A HARMONIC CHAIN

In this section we consider the *quantum* system corresponding to the classical model of the previous section. It is easily seen that the Hamiltonian has the form (hated quantities denoting operators-cf. paper I section 2 for the notation):

$$\hat{H} = \hat{H}_\Sigma + \hat{H}_R + \lambda \hat{H}_I =$$

$$= \hbar\Omega_0 \hat{a}^+\hat{a} + \sum_k \hbar\omega_k \hat{a}_k^+\hat{a}_k + \lambda\sum_k \hbar(\epsilon_k \hat{a}_k^+ + \epsilon_k^* \hat{a}_k)(\hat{a}+\hat{a}^+) \tag{3.1}$$

where

$$\hat{a} = \frac{1}{\sqrt{2\hbar}}(\sqrt{\Omega_0}\hat{q} + i\frac{\hat{p}}{\sqrt{\Omega_0}}) \qquad \hat{a}^+ = \frac{1}{\sqrt{2\hbar}}(\sqrt{\Omega_0}\hat{q} - i\frac{\hat{p}}{\sqrt{\Omega_0}}) \tag{3.2}$$

are the creation-annihilation operators for the oscillator and $\hat{a}_k^+, \hat{a}_k$ the corresponding quantities for the k-th bath oscillator, defined similarly(1), and Ω_0, ω_k *are nonegative.*

(1) Non-unit masses can be absorbed into $\hat{q}, \hat{p}$ by the same rescaling we did for the classical system.

In fact it is readily seen that any generalized Wigner transformation sending functions of $\hat{q}$ or $\hat{p}$ to functions of q or p respectively (i.e. (3.17) of paper I holds) maps (3.1) to (2.4)(see the comments following it), provided ϵ_k is real. For an infinite chain, in the thermodynamic limit we make the replacements

$$\omega_k \longrightarrow \omega \qquad \epsilon_k \longrightarrow \epsilon(\omega) \qquad \sum_k \longrightarrow \int d\omega \sigma(\omega)$$

$\sigma(\omega)$ being the spectral density of the chain. Consequently, if $\sigma = 1$, i.e. $\epsilon(k) = \epsilon(\omega_k)$, then (3.1) is mapped to (2.11) with $u^2(\omega) \leftrightarrow 4\epsilon^2(\omega)\omega\Omega_0$. A detailed comparison of the results of the two sections obviously requires $\omega > 0$ in the previous section and is discussed at the end of Appendix 1.

In the litterature, especially in quantum optics, the Hamiltonian (3.1) is often treated in the so-called rotating-wave approximation where the terms $\hat{a}_k^+\hat{a}^+$, $\hat{a}_k\hat{a}$ are neglected (see e.g. [8] p. 336 and eq. (6.2.39), [9] ch. 3 section 3b, implicity in [12] ch. 12, [22], [36] section 5.14). Sometimes such an approximation is made at the end of the derivation of the kinetic equation ([11] p. 743 eq. (3.1)). To derive a Markovian master equation for weakly coupled systems, such an approximation is not necessary and for our purpose not desirable, since the Hamiltonian should reduce to the classical one by a generalized Wigner transformation (see also Appendix 3). We indicate here the steps to apply the formalism of [5] that is, compute the generator Φ in Appendix 3, noting that $\hat{H}_\Sigma$ is separable (cf. [35] for rigorous estimates of the approach to equilibrium). Specifically we apply (5.8) of paper I to (3.1), with

$$\hat{H}_I = \hat{W}\hat{q} = \sum_k \hat{W}_k\hat{q} = \sum_k \hbar(\epsilon_k\hat{a}_k^+ + \epsilon_k^*\hat{a}_k^+)(\hat{a} + \hat{a}^+) \tag{3.3}$$

if $\nu = n\omega_0, \mathcal{F}_n$ are the eigenvalues and eigenprojections of $\frac{1}{\hbar}[\hat{H}_\Sigma, \cdot] = L_\Sigma$, $n \in \mathcal{Z}$, then clearly

$$\mathcal{F}_n\hat{q} = \sum_{m\in\mathcal{Z}} P_m\hat{q}P_{m-n}$$

where P_n are the eigenprojections of the number operator $\hat{N} = \hat{a}^+\hat{a}$. Using the well-known expression for its eigenvectors

$$\hat{N}|n >= n|n > \tag{3.4a}$$

$$\hat{a}|n >= \sqrt{n}\ |n-1> \qquad \hat{a}^+|n >= \sqrt{n+1}\ |n+1> \tag{3.4b}$$

a simple calculation gives

$$\mathcal{F}_n \hat{q} = (\hat{a}\delta_{n,-1} + \hat{a}^+ \delta_{n,1}) \tag{3.5}$$

On the other hand, similar expressions to (3.4) for the bath oscillator and

$$e^{iL_R s}\hat{A} = e^{i\frac{\hat{H}_R}{\hbar}s}\hat{A}e^{-i\frac{\hat{H}_R}{\hbar}s} \quad , \; L_R = \frac{1}{\hbar}[\hat{H}_R, \cdot]$$

give

$$e^{iL_R s}\hat{a}_k = e^{-i\omega_k s}\hat{a}_k \qquad\qquad e^{iL_R s}\hat{a}_k^+ = e^{i\omega_k s}\hat{a}_k^+ \tag{3.6}$$

Therefore by (3.3)

$$\hat{W}_k(s) \equiv e^{iL_R s}\hat{W}_k = \hbar(\epsilon_k e^{i\omega_k s}\hat{a}_k^+ + \epsilon_k^* e^{-i\omega_k s}\hat{a}_k^+) \tag{3.7}$$

As in the previous section, we assume the bath to be in canonical equilibrium

$$\hat{\rho}_R = \frac{e^{-\beta\hat{H}_R}}{Tr(e^{-\beta\hat{H}_R})} \tag{3.8}$$

A simple calculation using (3.1) gives

$$Tr(e^{-\beta\hat{H}_R}) = \prod_k Tr(e^{-\beta\hbar\omega_k\hat{a}_k^+\hat{a}_k}) = \prod_k (1 - e^{-\beta h\omega_k})^{-1}$$

$$< \hat{a}_k\hat{a}_{k'} >=< \hat{a}_k^+\hat{a}_{k'}^+ >= 0 \tag{3.9a}$$

$$< \hat{a}_k^+\hat{a}_{k'} >= \frac{\delta_{k,k'}}{e^{\beta\hbar\omega_k} - 1} \equiv \frac{n_k}{\hbar}\delta_{k,k'} \tag{3.9b}$$

$$< \hat{a}_{k'}\hat{a}_k^+ >= \left(\frac{n_k}{\hbar} + 1\right)\delta_{k,k'} \tag{3.9c}$$

where $< \hat{A} >= Tr(\hat{\rho}_R\hat{A})$ and the $\hbar$-dependence in (3.9) is shown explicitly so that n_k has a finite value $(\beta\omega_k)^{-1}$ in the classical limit. Using (3.9) we readily find

$$h(s) \equiv< \hat{W}^+\hat{W}(s) >= \hbar^2 \sum_k |\epsilon_k|^2 \left(e^{i\omega_k s}\left(\frac{n_k}{\hbar} + 1\right) + e^{-i\omega_k s}\frac{n_k}{\hbar}\right)$$

Using (A.1.3) we get

$$\tilde{h}(\omega) \equiv \int_{-\infty}^{+\infty} ds\; e^{i\omega s}h(s) =$$

$$2\pi\hbar^2 \sum_k |\epsilon_k|^2 \Big(\Big(\frac{n_k}{\hbar}+1\Big)\delta(\omega+\omega_k) + \frac{n_k}{\hbar}\delta(\omega-\omega_k) \Big) \tag{3.10a}$$

$$s(\omega) \equiv \Im \int_0^{+\infty} ds\, e^{i\omega s} h(s) = \hbar^2 \sum_k |\epsilon_k|^2 \Big(\Big(\frac{n_k}{\hbar}+1\Big)\frac{1}{\omega+\omega_k} + \frac{n_k}{\hbar}\frac{1}{\omega-\omega_k} \Big) \tag{3.10b}$$

We are now ready to obtain the kinetic equation for the density matrix $\hat\rho$ of the oscillator, by applying the above results to the general kinetic equation (5.8) of paper I, which for the present model reads:

$$\frac{\partial\hat\rho}{\partial t} = -\frac{1}{\hbar}[\hat H_\Sigma, \hat\rho] + \frac{i\lambda^2}{\hbar^2}\Big[\sum_n s(n\Omega_0)(\mathcal{F}_n\hat q)^+(\mathcal{F}_n\hat q),\ \hat\rho\Big] +$$

$$\frac{\lambda^2}{2\hbar^2}\sum_n \tilde h(n\Omega_0)\Big(\Big[(\mathcal{F}_n\hat q)\hat\rho,\ (\mathcal{F}_n\hat q)^+\Big] + \Big[\mathcal{F}_n\hat q,\ \hat\rho(\mathcal{F}_n\hat q)^+\Big]\Big) \tag{3.11}$$

In the thermodynamic limit of an infinite chain, applying (3.10), (3.5) to (3.11) using that $\omega > 0$ and making straightforward reductions we get ([26] ch. III)

$$\frac{\partial\hat\rho}{\partial t} = -\frac{i}{\hbar}\Big(1-\lambda^2\frac{\Delta(\Omega_0)}{\Omega_0}\Big)[\hat H_\Sigma, \hat\rho] +$$

$$+\pi\lambda^2|\epsilon(\Omega_0)|^2\sigma(\Omega_0)\Big(\frac{n(\Omega_0)}{\hbar}\Big([\hat a^+\hat\rho, \hat a] + [\hat a^+, \hat\rho\hat a]\Big) +$$

$$\Big(\frac{n(\Omega_0)}{\hbar}+1\Big)\Big([\hat a\hat\rho, \hat a^+] + [\hat a, \hat\rho\hat a^+]\Big)\Big) \tag{3.12}$$

$$\Delta(\Omega_0) = \int_0^{+\infty} d\omega \Big(\frac{1}{\omega-\Omega_0} + \frac{1}{\omega+\Omega_0}\Big)|\epsilon(\omega)|^2\sigma(\omega) \tag{3.13}$$

As already mentioned, (3.13) is identical with the well-known kinetic equation used in quantum optics and derived by making the rotating-wave approximation ([8] eq.(6.2.59), [9] eq.(3b.7) p.118, [11] eq.(3.1), [36] eq. (5.1.59)). On the other hand (3.13) is of the form (5.8) of paper I, with the following identifications:

$$\tilde h_{\alpha\beta}(\omega) \leftrightarrow \delta_{n,m} \quad , \quad n,\ m = 1,\ 2$$

$$\hat B_1 \leftrightarrow \hbar\gamma\sqrt{\frac{n(\Omega_0)}{\hbar}}\ \hat a^+ \qquad \hat B_2 \leftrightarrow \hbar\gamma\sqrt{\frac{n(\Omega_0)}{\hbar}+1}\ \hat a \tag{3.14a}$$

$$\gamma^2 = \pi|\epsilon(\Omega_0)|^2\sigma(\Omega_0) \tag{3.14b}$$

Therefore its phase-space representation via *any involutive* generalized Wigner transformation (cf.(3.16) of paper I) follows from (5.17) of paper I. From (3.2) we find the *Wigner transforms* of $\hat{B}_1, \hat{B}_2$

$$B_1 \equiv \gamma\sqrt{\frac{n(\Omega_0)}{2}}\Big(\sqrt{\Omega_0}q - \frac{ip}{\sqrt{\Omega_0}}\Big) \quad , \quad B_2 \equiv \gamma\sqrt{\frac{n(\Omega_0)+\hbar}{2}}\Big(\sqrt{\Omega_0}q + \frac{ip}{\sqrt{\Omega_0}}\Big)$$

and consequently their Fourier transforms (paper I, section2)

$$\tilde{B}_1(\eta,\xi) = 2\pi i\sqrt{\frac{n(\Omega_0)}{2}}\gamma\Big(\sqrt{\Omega_0}\delta(\xi)\delta'(\eta) - \frac{i}{\sqrt{\Omega_0}}\delta'(\xi)\delta(\eta)\Big)$$

$$\tilde{B}_2(\eta',\xi') = 2\pi i\sqrt{\frac{n(\Omega_0)+\hbar}{2}}\gamma\Big(\sqrt{\Omega_0}\delta(\xi')\delta'(\eta') + \frac{i}{\sqrt{\Omega_0}}\delta'(\xi')\delta(\eta')\Big)$$

Therefore (3.14) above and (5.18) of paper I finally give

$$B(\sigma',\sigma) = -\Big(\Omega_0 D_1\delta(\xi)\delta(\xi')\delta'(\eta)\delta'(\eta') + \frac{D_1}{\Omega_0}\delta'(\xi)\delta'(\xi')\delta(\eta)\delta(\eta')\Big)$$

$$+iD_2\Big(\delta(\xi)\delta'(\xi')\delta'(\eta)\delta(\eta') - \delta'(\xi)\delta(\xi')\delta(\eta)\delta'(\eta')\Big) \tag{3.15}$$

$$D_1 = 2\pi^2\gamma^2(2n(\Omega_0)+\hbar) \quad , \quad D_2 = 2\pi^2\gamma^2\hbar \tag{3.16}$$

Substitution of (3.15) in (5.17) of paper I gives after a straightforward calculation and with the aid of (2.12) of paper I, the phase-space representation of (3.12):

$$\frac{\partial\rho}{\partial t} = \frac{1}{i\hbar}\Big(1 - \lambda^2\frac{\Delta(\Omega_0)}{\Omega_0}\Big)[H_\Sigma,\rho] -$$

$$-\frac{2\lambda^2}{(2\pi)^3\hbar^2}\Re\Big(-\Omega_0 D_1\int d\sigma e^{i\sigma z}\tilde{\rho}(\sigma)\Big(\mu^2\xi^2 + i\mu\xi q - \mu\xi\frac{\partial\chi(\sigma)}{\partial\eta}\Big)$$

$$+\frac{D_1}{\Omega_0}\int d\sigma e^{i\sigma z}\tilde{\rho}(\sigma)\Big(-\mu^2\eta^2 + i\mu\eta p - \mu\eta\frac{\partial\chi(\sigma)}{\partial\xi}\Big)$$

$$+iD_2\int d\sigma e^{i\sigma z}\tilde{\rho}(\sigma)\Big(2\mu + i\mu(q\eta + p\xi) - \mu\eta\frac{\partial\chi(\sigma)}{\partial\eta} - \mu\xi\frac{\partial\chi(\sigma)}{\partial\xi}\Big)\Big) \tag{3.17}$$

where $\mu = \frac{i\hbar}{2}$, [,] is the bracket with respect to the $*_\Omega$- product, H_Σ is the classical Hamiltonian of the oscillator and the kernel of the generalized Wigner transformation is

$$\Omega(\sigma) = e^{\chi(\sigma)} \tag{3.18}$$

since Ω is an entire (analytic) function without zeros (cf. paper I, section 2). Eq.(3.18) follows then from the Weierstrass factor theorem for several variables (see e.g. [27]). Since the generalized Wigner transformation is assumed involutive, so that

$$\Omega(\sigma) = \Omega^*(-\sigma) \quad \Leftrightarrow \quad \chi(-\sigma) = \chi^*(\sigma)$$

(cf. (3.16) of paper I), we get

$$\tilde{\rho}(-\sigma)\frac{\partial\chi(-\sigma)}{\partial(-\eta)} = -\Big(\tilde{\rho}(\sigma)\frac{\partial\chi(\sigma)}{\partial\eta}\Big)^*$$

that is the Fourier transform of $\tilde{\rho}(\sigma)\frac{\partial\chi(\sigma)}{\partial\eta}$ (and for that matter, of $\tilde{\rho}(\sigma)\frac{\partial\chi(\sigma)}{\partial\xi}$) is an imaginary function hence it does not contribute to (3.17). The same is true for the second term in the first two integrals. Using this, simple reductions transform (3.17) with the aid of (3.16), to

$$\begin{aligned}\frac{\partial\rho}{\partial t} &= \frac{1}{i\hbar}\Big(1-\lambda^2\frac{\Delta(\Omega_0)}{\Omega_0}\Big)[H_\Sigma,\ \rho]+\\ &\quad+\frac{\lambda^2\gamma^2}{2}\Bigg[\frac{\partial}{\partial p}\Big(\Omega_0(n(\Omega_0)+\frac{\hbar}{2})\frac{\partial\rho}{\partial p}+p\rho+p\frac{\Psi}{2\pi}\odot\rho\Big)+\\ &\qquad+\frac{\partial}{\partial q}\Big(\frac{1}{\Omega_0}\Big(n(\Omega_0)+\frac{\hbar}{2}\Big)\frac{\partial\rho}{\partial q}+q\rho+q\frac{\Psi}{2\pi}\odot\rho\Big)\Bigg]\end{aligned} \tag{3.19}$$

where

$$\Psi(z) = \frac{1}{2\pi}\int e^{i\sigma z}\chi(\sigma)d\sigma \tag{3.20}$$

and $\odot$ is the convolution product. This is the desired phase-space equation for the quantum oscillator, obtained via *any* involutive generalized Wigner transformation. It is now trivial to obtain the classical limit of (3.19) (or equivalently of (3.12)), since all quantities, except Ψ are $\hbar$-independent. However assuming that $\frac{1}{i\hbar}[\ ,\ ]$ is a deformation of the Poisson bracket, proposition 3.2 of paper I and (3.18) imply that if $(\hat{q},\hat{p})$ is mapped to (q,p) then $\lim_{\hbar\to 0^+}\Psi(z)=0$, hence if $\lim_{\hbar\to 0^+}\rho=\rho_0$, then

$$\begin{aligned}\frac{\partial\rho_0}{\partial t} &= \Big(1-\lambda^2\frac{\Delta(\Omega_0)}{\Omega_0}\Big)\{H_\Sigma,\rho_0\}+\\ &\quad+\frac{\lambda^2\gamma^2}{2}\Big(\frac{\partial}{\partial p}\Big(\Omega_0 n(\Omega_0)\frac{\partial\rho_0}{\partial p}+p\rho_0\Big)+\frac{\partial}{\partial q}\Big(\frac{1}{\Omega_0}n(\Omega_0)\frac{\partial\rho_0}{\partial q}+q\rho_0\Big)\Big)\end{aligned} \tag{3.21}$$

Taking account of (3.9b), (3.14d) and the remarks at the end of Appendix 1 we see that (3.21) with $\sigma = 1$ *is identical with the classical equation (2.26)* since $u^2(\omega), \omega > 0$, corresponds to $4\epsilon(\omega)\omega\Omega_0$ as mentioned at the beginning of this section. When $\sigma \neq 1$ the comparison should be made with the kinetic equation following by using the Hamiltonian (2.8) instead of (2.11) in section 2. However in view of the comments made at the end of Appendix 1, once again the two equations turn to be identical.

4. KINETIC EQUATION FOR MORE GENERAL HARMONIC OSCILLATOR MODELS

In the previous section we obtained the kinetic equation for a quantum harmonic oscillator, weakly coupled to an equilibrium bath of other oscillators, eq(3.12) within the context of the general formalism of [5]. Moreover its phase-space repsesentation via an arbitrary involutive generalized Wigner transformation was obtained, eq.(3.19), from which the uniqueness of its classical limit,eq.(3.21), follows. Direct application of (3.2) to (3.12) shows that the latter is a special case of the following more general equation

$$\begin{aligned}\frac{\partial \hat{\rho}}{\partial t} = -\frac{i}{\hbar}[\hat{H}_\Sigma, \hat{\rho}] - \frac{i}{\hbar}(\Lambda + \kappa)[\hat{q}, [\hat{\rho}, \hat{p}]_+] + \frac{i}{\hbar}(\Lambda - \kappa)[\hat{p}, [\hat{\rho}, \hat{q}]_+] \\ -\frac{D_1}{\hbar^2}[\hat{q}, [\hat{q}, \hat{\rho}]] - \frac{D_2}{\hbar^2}[\hat{p}, [\hat{p}, \hat{\rho}]] + \frac{D}{\hbar^2}([\hat{q}, [\hat{p}, \hat{\rho}]] + [\hat{p}, [\hat{q}, \hat{\rho}]]) \qquad (4.1)\end{aligned}$$

with $\hat{H}_\Sigma$ given by (3.1).

This equation has been considered in the litterature as a master equation in quantum optics, describing an electromagnetic field mode interacting with an equilibrium bath of bosons. It also includes many other master equations used to study open systems in heavy ion collisions ([13] section 3,[14] section 1,2 and references therein). As it will be seen, it is of the Lindblad type. In fact, it follows from it (eq.(5.10$''$)) of paper I) assuming that as functions of $\hat{q}, \hat{p}$, its non dissipative part is quadratic and the operators appearing in the dissipative part, are linear ([13] section 3, eq.(3.6)).

In [14] its phase-space representation by means of Wigner's transformation or its generalizations corresponding to antinormal and normal ordering of operators (Glauber repsesentation) have been given, showing that it is an equation of the Fokker-Planck type, i.e. with nonegative-definite 2nd order term, provided that

$$D_1, D_2 > 0 \quad D_1 D_2 - D^2 \geq \frac{\hbar^2 \Lambda^2}{4} \qquad (4.2)$$

The formalism of paper I allows us to generalize this result obtaining the phase-space analogue of (4.1) for *any involutive generalized Wigner transformation.* The classical limit of (4.1) is then a simple matter, and as for the special case of section 3, it turns out to be unique.

Rearranging the second term on the r.h.s. of (4.1), we get

$$\frac{\partial\hat\rho}{\partial t} = -\frac{i}{\hbar}[\hat H_\Sigma + \kappa[\hat q,\hat p]_+,\hat\rho] -$$

$$-\frac{1}{\hbar^2}(D_1[\hat q,[\hat q,\hat\rho]] + D_2[\hat p,[\hat p,\hat\rho]] - D([\hat q,[\hat p,\hat\rho]] + [\hat p,[\hat q,\hat\rho]])$$

$$-\frac{i\Lambda}{\hbar}\Big([\hat q,[\hat p,\hat\rho]_+] - [\hat p,[\hat q,\hat\rho]_+]\Big) \tag{4.1$'$}$$

Comparison with (5.5) of paper I shows that (4.1$'$) is of the same form with the identifications

$$\frac{\lambda^2}{2}\tilde h^q_{\alpha\beta} = \begin{pmatrix} D_1 & -D \\ -D & D_2 \end{pmatrix}, \qquad \frac{\lambda^2}{2}\tilde g^q_{\alpha\beta} = \begin{pmatrix} 0 & \Lambda \\ -\Lambda & 0 \end{pmatrix}$$

$$\hat V_1 = \hat V_1^+ = \hat q \quad , \quad V_2 = \hat V_2^+ = \hat p$$

Therefore it can be put in the form (5.8) of paper I, hence it is of the Lindblad type, (eq.(5.10$''$) of paper I) with $\hat V_1, \hat V_2$ as above and

$$\frac{\lambda^2}{2\hbar^2}\tilde h_{\alpha\beta} = \frac{1}{\hbar^2}\begin{pmatrix} D_1 & \frac{i\hbar\Lambda}{2} - D \\ -\frac{i\hbar\Lambda}{2} - D & D_2 \end{pmatrix} \tag{4.3}$$

provided the latter is nonegative-definite, from which (4.2) follows). Hence its phase-space representation is given by (5.17) of paper I with

$$B_1 \equiv V_1 \equiv \Omega_w^{-1}(\hat V_1) = q, \quad B_2 \equiv V_2 \equiv \Omega_w^{-1}(\hat V_2) = p$$

and Ω_w^{-1} indicating the Wigner transformation (c.f. section 2.1 of paper I). Consequently their Fourier transforms are

$$\tilde B_1(\eta',\xi') = 2\pi i\delta'(\eta')\delta(\xi') \quad \tilde B_2(\eta,\xi) = 2\pi i\delta(\eta)\delta'(\xi) \tag{4.4}$$

so that with the usual notation $\sigma = (\eta,\xi)$, $z = (q,p)$ etc, (4.3), (4.4) substituted in (5.18) of paper I, give

$$\frac{\lambda^2}{2}B(\sigma,\sigma') \equiv \sum_{n,m=1}^{2} \tilde V_n^*(\sigma')\tilde h_{nm}\tilde V_m(\sigma) =$$

$$= -(2\pi)^2 \Bigg(D_1 \delta'(\eta)\delta'(\eta')\delta(\xi)\delta(\xi') + D_2 \delta(\eta)\delta(\eta')\delta'(\xi)\delta'(\xi') +$$

$$+ (\frac{i\hbar\Lambda}{2} - D)\delta'(\eta')\delta(\eta)\delta(\xi')\delta'(\xi) - (\frac{i\hbar\Lambda}{2} + D)\delta'(\eta)\delta(\eta')\delta(\xi)\delta'(\xi') \Bigg) \qquad (4.5)$$

Substituting (4.5) to (5.17) of paper I gives the desired kinetic equation in phase-space. The calculations are tedious but not difficult in principle and are given in Appendix 2. The result is

$$\frac{\partial \rho}{\partial t} = D_1 \frac{\partial^2 \rho}{\partial p^2} + D_2 \frac{\partial^2 \rho}{\partial q^2} + 2D \frac{\partial^2 \rho}{\partial p \partial q} +$$

$$+ \frac{\partial}{\partial p}[(\Omega_0^2 q + (\Lambda + \kappa)p)(\rho + \frac{\Psi}{2\pi} \odot \rho)] + \frac{\partial}{\partial q}[(-\frac{p}{m} + (\Lambda - \kappa)q)(\rho + \frac{\Psi}{2\pi} \odot \rho)] \quad (4.6)$$

where as in the previous section Ψ is given by (3.18), (3.20). Therefore it is clear that if the generalized Moyal bracket defined by the generalized Wigner transformation is a deformation of the Poisson bracket, and $(\hat{q}, \hat{p})$ are mapped to (q, p) then, as in the previous section $\lim_{\hbar \to 0} \Psi(q, p) = 0$ and (4.6) has the *unique* classical limit ($\lim_{\hbar \to 0} \rho \equiv \rho_0$)

$$\frac{\partial \rho_0}{\partial t} = \{H_\Sigma, \rho_0\} +$$

$$+ \frac{\partial}{\partial p}(D_1 \frac{\partial \rho}{\partial p} + D \frac{\partial \rho}{\partial q} + (\Lambda + \kappa)p\rho) + \frac{\partial}{\partial q}(D_2 \frac{\partial \rho}{\partial q} + D \frac{\partial \rho}{\partial p} + (\Lambda - \kappa)q\rho) \qquad (4.7)$$

where H_Σ is the classical hamiltonian for the oscillator, (2.4a)

To recover known results, we explicitly calculate the convolution terms in (4.6) for

$$\Omega(\eta, \xi) = e^{a \frac{\hbar}{4}(\Omega_0^2 \xi^2 + \frac{\eta^2}{\Omega_0})} \quad a \in \mathcal{R} \qquad (4.8)$$

The following cases are included in (4.8) ([28])

$a = 0$: *Weyl ordering*

$a = -1$: *normal ordering (Glauber representation)*

$a = 1$: *antinormal ordering*

From (4.8), (3.20) we find

$$\Psi(q, p) = -\frac{2\pi\lambda\hbar}{4\Omega_0}(\delta(p)\delta''(q) + \Omega_0^2 \delta(q)\delta''(p))$$

Since $x\delta''(x) = -2\delta'(x), x\delta(x) = 0$ a direct calculation gives

$$\frac{\partial}{\partial p}[(\Omega_0^2 q + (\Lambda+\kappa)p)(\frac{\Psi}{2\pi}\odot\rho)] + \frac{\partial}{\partial q}[(-\frac{p}{m} + (\Lambda-\kappa)q)(\frac{\Psi}{2\pi}\odot\rho)] =$$

$$= -\frac{\hbar a}{2}\Omega_0(\Lambda+\kappa)\frac{\partial^2\rho}{\partial p^2} - \frac{\hbar a}{2\Omega_0}(\Lambda-\kappa)\frac{\partial^2\rho}{\partial q^2}$$

Substituting this in (4.6), we finally get

$$\frac{\partial\rho}{\partial t} = (D_1 - a\hbar\frac{(\Lambda+\kappa)}{2}\Omega_0)\frac{\partial^2\rho}{\partial p^2} + (D_2 - a\hbar\frac{(\Lambda-\kappa)}{2\Omega_0})\frac{\partial^2\rho}{\partial q^2} + 2D\frac{\partial^2\rho}{\partial q\partial p}$$

$$+\frac{\partial}{\partial p}[(\Omega_0^2 q + (\Lambda+\kappa)p)\rho] + \frac{\partial}{\partial q}[(-\frac{p}{m} + (\Lambda-\kappa)q)\rho] \tag{4.9}$$

Introducing the variables

$$x_1 = \sqrt{\frac{\Omega_0}{2\hbar}}q \qquad x_2 = \frac{p}{\sqrt{2\hbar\Omega_0}}$$

So that $x = x_1 + ix_2$ are the eigenvalues of $\hat{a}$ corresponding to coherent states (e.g. [8] p.107-108), (4.9) becomes

$$\frac{\partial\rho}{\partial t} = \left(\frac{D_1}{2h\Omega_0} - a\frac{(\Lambda+\kappa)}{4}\right)\frac{\partial^2\rho}{\partial x_2^2} + \left(\frac{D_2\Omega_0}{2\hbar} - a\frac{(\Lambda-\kappa)}{4}\right)\frac{\partial^2\rho}{\partial x_1^2} + \frac{D}{\hbar}\frac{\partial^2\rho}{\partial x_1\partial x_2}$$

$$\frac{\partial}{\partial x_1}[((\Lambda-\kappa)x_1 - \omega x_2)\rho] + \frac{\partial}{\partial x_2}[((\Lambda+\kappa)x_2 + \omega x_1)\rho] \tag{4.9$'$}$$

For $a = 0, \pm 1$ this is identical with eq. (4.5) and Table III of [14], see also [13] eq.(5.19) for the case $a = 0$

In the present paper the general formalism of paper I for the phase-space representation of quantum kinetic equations, has been applied to models of a harmonic oscillator damped by its interaction with an equilibrium bath of other harmonic oscillators. These equations are of the Lindblad type, but it should be emphasized that *the methods of paper I allow for a phase-space representation of any kinetic equation the evolution operator of which is formed by algebraic operations between quantum operators.* Other applications, as well as some more fundamental problems of kinetic theory, already briefly discussed in paper I section 1 and [29] section 4, will be examined in the third paper of this series.

APPENDIX 1

Here we derive eqs (2.22), (2.23) by using simple functional integration (see e.g. [24]): By (2.18), H_R is quadratic in ϕ, π hence by the formula

$$\frac{\delta}{\delta\phi(\omega)}\left(e^{-\beta H_R}\right) = -\beta\frac{\delta H_R}{\delta\phi(\omega)}\ e^{-\beta H_R}$$

and similarly for $\pi(\omega)$ we get

$$\int \delta\phi\delta\pi\ \phi(\omega)\phi(\omega')\rho_R = -\int \delta\phi\delta\pi\ \frac{\phi(\omega)}{\beta\omega'^2}\frac{\delta\rho_R}{\delta\phi(\omega')} =$$

$$= \frac{1}{\beta\omega'^2}\int \delta\phi\delta\pi\ \frac{\delta\phi(\omega)}{\delta\phi(\omega')}\rho_R = \frac{1}{\beta\omega'^2}\delta(\omega-\omega')$$

where we have used integration by parts and have taken into account that

$$\int \delta\phi\delta\pi\ \frac{\delta}{\delta\phi}\left(\phi^2\rho_R\right) = 0$$

Similar calculations give

$$\int \delta\phi\delta\pi\ \pi(\omega)\pi(\omega')\rho_R = \frac{1}{\beta}\delta(\omega-\omega')$$

$$\int \delta\phi\delta\pi\ \pi(\omega)\phi(\omega')\rho_R = 0$$

$$\int \delta\phi\delta\pi\ \phi(\omega)\phi(\omega') = \frac{1}{\beta\omega^2}\delta(\omega-\omega')$$

Using this and (2.14b) in (2.17a) we find

$$h(s) = \int d\omega\frac{u^2(\omega)}{\beta\omega^2}\cos\omega s \qquad (A.1.1)$$

Similarly, using

$$\frac{\delta\phi(\omega)}{\delta\phi(\omega')} = \frac{\delta\pi(\omega)}{\delta\pi(\omega')} = \delta(\omega-\omega'), \qquad \frac{\delta\phi(\omega)}{\delta\pi(\omega')} = 0$$

and that $\int \delta\phi\delta\pi\ \rho_R = 1$, we find from (2.17b) that

$$g(s) = \int d\omega\frac{u^2(\omega)}{\omega}\sin\omega s \qquad (A.1.2)$$

Using the well-known formulas

$$\int_{-\infty}^{+\infty} ds\ e^{ias} = 2\pi\delta(a), \qquad \int_{0}^{+\infty} ds\ e^{ias} = \pi\delta(a) + \frac{i}{a}$$

we get

$$\int_{-\infty}^{+\infty} ds\ e^{ias} \cos\omega s = \pi\Big(\delta(a-\omega) + \delta(a+\omega)\Big) \tag{A.1.3a}$$

$$\int_{-\infty}^{+\infty} ds\ e^{ias} \sin\omega s = \pi\Big(\delta(a-\omega) - \delta(a+\omega)\Big) \tag{A.1.3b}$$

$$\int_{0}^{+\infty} ds\ e^{ias} \cos\omega s = \frac{\pi}{2}\Big(\delta(a-\omega)+\delta(a+\omega)\Big)+\frac{ia}{a^2-\omega^2} \tag{A.1.3c}$$

$$\int_{0}^{+\infty} ds\ e^{ias} \sin\omega s = \frac{i\pi}{2}\Big(\delta(a-\omega)-\delta(a+\omega)\Big)-\frac{\omega}{a^2-\omega^2} \tag{A.1.3d}$$

Substituting (A.1.1) in (2.17a) and using (A.1.3a) we get

$$\tilde{h}(a) = \pi \int \Big(\delta(a-\omega)+\delta(a+\omega)\Big)\frac{u^2(\omega)}{\beta\omega^2} d\omega \tag{A.1.4}$$

which gives the first of (2.22). The expressions for $\tilde{g}, \bar{h}, \bar{g}$ in (2.22), (2.23′) are similarly obtained. We may notice here that if instead of the Hamiltonian (2.11) we consider (2.8), then nothing changes in the preceding calculations as well as those of section 2, except that the integration in (A.1.1), (A.1.2) are with respect to θ. Hence in (A.1.3) ω is replaced by $\omega(\theta)$ and consequently by a change of variables $\theta \longrightarrow \omega(\theta)$, (A.1.4) gives

$$\tilde{h}(a) = \frac{2\pi\Big(u(\theta(a))\Big)^2}{a^2}\sigma(a) \qquad , \qquad \sigma(\omega) = \frac{d\theta}{d\omega}$$

provided $\omega(\theta)$ is an invertible, even function of θ (notice that because of (2.10), $\omega(\theta)$ is necessarily either even or odd). It is now clear that in the kinetic equation (2.26) (and in (2.25) as well), u^2 is replaced by $u^2\sigma$, which has already been remarked in section 2.

If the spectral variable ω in (2.11) is nonegative then $u(\omega)$ can be extended by putting $u(\omega)=0$, for $\omega\leq 0$ and all calculations remain unaltered except that

(i) in (A.1.4) (and the corresponding equation for $\tilde{g}$), the one δ-function does not contribute
(ii) by (2.22), $\tilde{h}(\omega_{-1})=\tilde{g}(\omega_{-1})=0$, hence the 2nd term of (2.21) does not contribute to (2.15).

Then it is readily checked that these changes imply a factor $\frac{1}{4}$ on the r.h.s. of (2.24). The case $\omega>0$ is used in section 3 to show explicitly that (2.26) thus modified, is the classical limit of the kinetic equation for the corresponding quantum system.

APPENDIX 2

Here we derive (4.6). For the dissipative part we substitute (4.5) to eq(5.17) of paper I. Specifically we compute

$$-\frac{2\lambda^2}{(2\pi)^3\hbar^2}\Re\int d\sigma d\sigma' d\sigma''\frac{e^{i(\sigma+\sigma'+\sigma'')}}{\Omega(\sigma+\sigma'+\sigma'')}B(\sigma',\sigma)$$

$$\tilde{\rho}_w(\sigma'')e^{\mu(\sigma'\wedge\sigma'')}\sinh\mu\Big(\sigma\wedge(\sigma'+\sigma'')\Big)$$

where $\tilde{\rho}_w$ is the Fourier transform of the Wigner transform of $\hat{\rho}$, $\sigma=(\eta,\xi)$, $\sigma\wedge\sigma'=\eta\xi'-\eta'\xi$ (see section 2 of paper I for the notation). Substitution of $B(\sigma',\sigma)$ from (4.5) and straightforward integrations of the δ-functions give

$$\frac{4}{2\pi\hbar^2}\Re\int d\sigma e^{i\sigma z}\tilde{\rho}(\sigma)\Big[D_1\Big(\mu^2\xi^2+i\mu\xi q-\mu\xi\partial_\eta\chi(\sigma)\Big)+$$

$$+D_2\Big(\mu^2\eta^2-i\mu\eta p+\mu\eta\partial_\eta\chi(\sigma)\Big)-\Big(\frac{i\hbar\Lambda}{2}-D\Big)\Big(\mu+i\mu q\eta+\mu^2\eta\xi-\mu\eta\partial_\eta\chi(\sigma)\Big)$$

$$-\Big(\frac{i\hbar\Lambda}{2}+D\Big)\Big(\mu+i\mu p\xi-\mu^2\eta\xi-\mu\xi\partial_\xi\chi(\sigma)\Big)\Big]$$

where we used (2.12) of paper I and (3.18). It is not difficult to see that this is $\left(\mu=\frac{i\hbar}{2}\right)$

$$D_1\frac{\partial^2\rho}{\partial p^2}+D_2\frac{\partial^2\rho}{\partial q^2}+2D\frac{\partial^2\rho}{\partial q\partial p}+\Lambda\frac{\partial}{\partial p}\Big[p\Big(\rho+\frac{\Psi}{2\pi}\odot\rho\Big)\Big]+\Lambda\frac{\partial}{\partial q}\Big[q\Big(\rho+\frac{\Psi}{2\pi}\odot\rho\Big)\Big]\quad (A.2.1)$$

For the nondissipative term of (4.1′) we proceed as follows: Since $\hat{H}_\Sigma$ is given by (3.1), (3.2) its Wigner transform is the classical Hamiltonian $H_\Sigma = \frac{p^2}{2} + \Omega_0^2 q^2$ hence its Fourier transform is

$$\tilde{H}_\Sigma = -2\pi\Big(\frac{\delta(\eta)\delta''(\xi)}{2} + \Omega_0^2\delta''(\eta)\delta(\xi)\Big)$$

Substituting this in (5.12b) of paper I we get after some simple reductions and using (2.12) of paper I:

$$[H_\Sigma, \rho] = \frac{1}{2\pi}\Bigg(-\frac{1}{2}\int d\sigma e^{i\sigma z}\tilde{\rho}(\sigma)\Big(4i\mu\eta p - 4\mu\eta\partial_\xi\chi(\sigma)\Big)$$

$$+\frac{\Omega_0^2}{2}\int d\sigma e^{i\sigma z}\tilde{\rho}(\sigma)\Big(4i\mu\xi q - 4\mu\xi\partial_\eta\chi(\sigma)\Big)\Bigg)$$

This is easily transformed to

$$-\frac{i}{\hbar}[H_\Sigma, \rho] = \{H_\Sigma, \rho\} + \Omega^2\frac{\partial}{\partial p}\Big(q\frac{\Psi}{2\pi}\odot\rho\Big) - \frac{\partial}{\partial q}\Big(p\frac{\Psi}{2\pi}\odot\rho\Big) \qquad (A.2.2)$$

Finally, since $(\hat{q}, \hat{p})$ is mapped to (q, p) by the Wigner transformation, a similar calculation using (5.15b) of paper I yields

$$-\frac{i}{\hbar}\kappa\Big[[q,p]_+, \rho\Big] = \kappa\Bigg(p\frac{\partial\rho}{\partial p} - q\frac{\partial\rho}{\partial q} - \frac{\partial}{\partial q}\Big(q\frac{\Psi}{2\pi}\odot\rho\Big) + \frac{\partial}{\partial p}\Big(p\frac{\Psi}{2\pi}\odot\rho\Big)\Bigg) \qquad (A.2.3)$$

Combining (A.2.1-3) gives (4.6).

APPENDIX 3

In sections 2, 3 we derive kinetic equations for a harmonic oscillator, weakly-coupled to a harmonic chain, by applying the general formal theory of open systems Σ weakly coupled to equilibrium baths R, developped in [5] (see also paper I, section 4.2). The starting point of this formalism is the solution of the Liouville (or von Neumann) equation for the total system, for positive times. This implies that the corresponding resolvent operator is analytic on the complex upper half-plane and has a cut along (part of) the real axis, which contains its necessarily existing continuous spectrum. Assuming that the restriction of this resolvent to the state space of the open system admits a meromorphic analytic continuation through the cut in the lower half-plane,

it can be shown that the corresponding exponentially decaying contributions to the solution of the exact dynamics satisfy a Markovian equation. Moreover, its generator can be given a representation in the subspace of the open system under certain assumptions which will not be discussed here (cf. [5] section 2). This generator, Φ say, has been computed to second order in the coupling parameter of the open system with the bath ([5] section 3) and it has been shown to be identical to that proposed by Davies ([3] eq.(2.17). See also [32]). It is used in this paper for the particular system studied. Specifically if the Liouville operator L of the total system is splitted as (see (4.6) of paper I)

$$L = L_\Sigma + L_R + \lambda L_I \equiv L_0 + \lambda L_I$$

and P is the projection onto the subspace of the open system (see (4.7) of paper I), then to second order in the coupling parameter λ, Φ is ([5] eq.(3.11′))

$$\Phi = -iPLP - \lambda^2 \lim_{T \longrightarrow +\infty} \frac{1}{2T} \int_{-T}^{T} dt \int_{0}^{+\infty} ds \; P e^{iL_0(s+t)} L_I Q e^{-iL_0 s} Q L_I e^{-iL_0 t} P$$

where Q is the complementary projection to P. Here it has been assumed that P commutes with L_0, i.e. for $\lambda = 0$, Σ evolves independently of R which stays in equilibrium, and that L_Σ has a point spectrum.

On the other hand, projecting Liouville's (or von Neumann's) equation to the P subspace, the so-called Generalized Master Equation (GME) results, which under the same assumptions, lead to *a quite different generator*, Θ say, ([33] section 2, eq.(2.10) and [5] eq.(3.7′))

$$\Theta = -iPLP - \lambda^2 \int_{0}^{+\infty} ds \; P L_I Q e^{-iL_0 s} Q L_I e^{iL_0 s} P$$

Notice that

$$\Phi = \lim_{T \longrightarrow +\infty} \frac{1}{2T} \int_{-T}^{T} dt \; e^{iL_0 t} \Theta e^{-iL_0 t}$$

Θ is essentially the generator that lead to kinetic equations with serious defects (nonconservation of positivity of the states and for that matter, no H-theorem), like (2.27) for classical systems (see the discussion in [5] section 5) and which call for additional assumptions, like the rotating-wave approximation mentioned in section 3, in order to get rid of these defects.

For a separable interaction Hamiltonian, Φ takes the form (4.22′) of [5], which for the model of section 2 gives (2.15), whereas for its quantum analogue it gives (3.11) (notice that $PL_IP = 0$ in our case).

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